Notes on SL(2) conformal fields theories. Exact solution and applications *.

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In these notes I briefly outline $SL(2)$ degenerate conformal field theories and their application to some related models, namely 2d gravity and $N = 2$ discrete superconformal series.

1. Introduction

Since the seminal work of Belavin, Polyakov and Zamolodchikov [1], where a general approach to two-dimensional conformal field theories was proposed, there has been much progress in understanding these theories. However the full solutions were found only for relatively few theories. The most famous examples are the diagonal minimal models and $SU(2)$ WZW models [2,3]. One motivation for my research was to extend this set by solving $SL(2)$ degenerate conformal field theories. These theories contain, as a subclass, $SU(2)$ models. Another motivation was to try to get information on more complicated models using a progress with $SL(2)$ ones.

The outline of these notes is as follows.
In section two I give a more formal discussion of the basic points relevant for $SL(2)$ degenerate conformal field theories. Next, in sections three and four, I present explicit examples of application of the results described in section two to 2d gravity and to some $N = 2$ discrete superconformal series. Finally, in section five I offer my conclusions and mention a few important problems.

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2. SL(2) degenerate conformal field theories

The theories have $\hat{s}l_2 \oplus \hat{s}l_2$ algebra as the symmetry algebra. The commutation relations for the holomorphic (antiholomorphic) part are given by

$$[J_n^\alpha, J_m^\beta] = f_\gamma^{\alpha\beta} J_{n+m}^\gamma + \frac{k}{2} n\, g^{\alpha\beta} \delta_{n+m} \quad , \tag{1}$$

where k is the level, $g^{\alpha\beta}$ is the Killing metric of sl_2 and $f_\gamma^{\alpha\beta}$ are its structure constants.

The complete system of states (Hilbert space) involved in the theory can be decomposed as

$$\mathcal{H} = \oplus_{\{j,\bar{j}\}} \Phi^{[j]} \otimes \Phi^{[\bar{j}]} \quad . \tag{2}$$

Here $\Phi^{[j]}$ is a representation of $\hat{s}l_2$.

I will only consider the diagonal embedding the Hilbert space into a tensor product of two holomorphic spaces of states in what follows. Such models are known in the literature as "A" series. Due to this reason I will suppress the $\bar{j}$-dependence as well as $\bar{\Delta}\,, \bar{h}$ etc below.

Let me also restrict to the case when $\Phi^{[j]}$ are the highest weight representations of $\hat{s}l_2$. In this case all reducible representations are known [4], namely, they are given by the Kac-Kazhdan set

$$j^+_{n.m} = \frac{1-n}{2}(k+2) + \frac{m-1}{2} \quad ,$$
$$j^-_{n.m} = \frac{n}{2}(k+2) - \frac{m+1}{2} \quad , \tag{3}$$

with $k \in \mathbf{C}\,,\ \{n, m\} \in \mathbf{N}$. Note that the unitary representations are given by $j^+_{1.m}$ with the integer level k.

In general, given a representation of a symmetry algebra, to define a field theory, one needs a construction attaching representation to a point on a curve. In the particular case at hand, a representation should be attached to a pair. The first parameter is a point on the Riemann surface. As to the second, it can be taken as an isotopic coordinate. From a mathematical point of view, this has been established in [5]. However, this has a very simple physical interpretation. Since Δ(conformal dimension) is quadratic in j(weight) one has to introduce additional parameters in order to define OP algebra of physical fields unambiguously otherwise it is defined up to $j=-j-1$ identification. There is no problem for the unitary case; this is, however, not the case for a general j given by (3).

It is surprising that the unitary models were solved by Fateev and Zamolodchikov using this improved construction attaching representation to a point [3]. So it seems very natural to postulate some basic OP expansions derived in that work as defining relations for $SL(2)$ conformal field theories whose primary fields are parametrized by the set (3). This was done in [6]. I will call such theories as the degenerate $SL(2)$ conformal field theories.

Define $x(\bar{x})$-dependent generators of $\hat{sl}_2$ as

$$J_n^-(x)=J_n^-\ ,\ J_n^0(x)=J_n^0+xJ_n^- \quad ,$$
$$J_n^+(x)=J_n^+-2xJ_n^0-x^2J_n^- \quad . \tag{4}$$

Here x is an isotopic coordinate. It is easy to verify that $J_n^\alpha(x)$ have the same commutation relations as J_n^α (see (1)), i.e. they form the Kac-Moody algebra. Next I proceed along the standard lines. Introducing the highest weight representations $\Phi^{[j]}(x)\otimes\Phi^{[j]}(\bar{x})$ one automatically generates the primary fields $\Phi^j(x,\bar{x},z,\bar{z})$ together with all their descendants. In above, z is a point on the sphere. The OP expansion for such primaries is given by

$$\Phi^{j_1}(x_1,\bar{x}_1,z_1,\bar{z}_1)\Phi^{j_2}(x_2,\bar{x}_2,z_2,\bar{z}_2)=$$
$$\sum_{j_3}\frac{|x_{12}|^{2(j_1+j_2-j_3)}}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}}C^{j_1j_2}_{j_3}\Phi^{j_3}(x_2,\bar{x}_2,z_2,\bar{z}_2)\,, \tag{5}$$

with $\Delta=j(j+1)/(k+2)$. The coefficients $C^{j_1j_2}_{j_3}$ are called the structure constants of the Operator Product algebra. It is evident that the isotopic coordinates provides the well-defined OP algebra.

The two and three point functions of the primary fields are defined from $SL(2)$ invariances. As to the others, they are found from the Knizhnik-Zamolodchikov equations. General four point functions were derived in [6]. Moreover in this work I wrote down the structure constants of the OP algebra (5). So the SL(2) degenerate conformal field theories were solved.

From the set (3) it is worth to distinguish the so-called admissible representations [7], which correspond to the rational level k, namely, $k+2=p/q$ with the coprime integers p and q. In this case there is a symmetry $j^-_{n.m}=j^+_{q-n+1,p-m}$ which allows one to reduce the primaries parametrized by $j^-_{n.m}$ to the ones parametrized by $j^+_{n.m}$. The OP algebra is closed in the grid $1\leq n_i\leq q\ ,\ 1\leq m_i\leq p-1$. The corresponding fusion rules are given by

$$|n_{12}|+1\leq n_3\leq min\begin{pmatrix} n_1+n_2-1\\ 2q-n_1-n_2+1\end{pmatrix}\ , \tag{6}$$
$$|m_{12}|+1\leq m_3\leq min\begin{pmatrix} m_1+m_2-1\\ 2p-m_1-m_2-1\end{pmatrix}\ ,$$

with the following steps $\Delta n_3=1\ ,\ \Delta m_3=2$.
This fusion rules were first found in [3,8] from the differential equations for the conformal blocks. They reveal the quantum group structure $(U_qosp(2/1),U_qsl(2))$ of the models [9].

To complete the story on $SL(2)$, I would like to refer to recent works [10].

3. 2d gravity coupled to $c\leq 1$ matter in the Polyakov light-cone gauge

This section attempts to briefly describe an application of the results obtained in section two to 2d gravity (see [11] for details).

Since the seminal works of Polyakov, Knizhnik and Zamolodchikov [12], there has been much progress in understanding the continuum fields theory approach to 2d gravity. The majority of efforts has been devoted to the study of coupling of conformal matter to gravity in the conformal gauge. The reason why it is useful lies in the fact that it is the standard gauge and its properties on the Riemann surfaces are well known. At the

same time, the properties of the Polyakov gauge are little known which restricts the applications of such a gauge. However it is turned out that the $SL(2)/SL(2)$ topological model reformulated in terms of the previous section provides a way to investigate problems in the light-cone gauge. Such model has $\hat{s}l_2 \oplus \hat{s}l_2 \oplus \hat{s}l_2$ algebra as the symmetry algebra [13]. The last term is a contribution of the first order fermionic system (ghosts) of weights (1.0). The levels are given by

$$k_1 = k \quad , \quad k_2 = -k-4 \quad , \quad k_3 = 4 \quad . \tag{7}$$

The physical fields (holomorphic part) at ghost number zero can be written as

$$\Phi^{j_1 \cdot j_2}(x,\bar{x},z) = \Phi^{j_1}(x,z)\Phi^{j_2}(\bar{x},z) \quad , \tag{8}$$

where Φ^j are the primaries of $\hat{s}l_2$.

The idea that the $SL(2)/SL(2)$ model is connected to the minimal models coupled to gravity was put forward in ref.[14]. This discusses mainly the conformal gauge. Let me now show how it works in my framework (with the isotopic coordinates). Setting $x = \bar{x} = z$ and $j_2 = -j_1 - 1$ with j_1 defined in (3) one immediately obtains the minimal model coupled to gravity, more correctly only its *holomorphic* sector, in the conformal gauge. It is surprising that there exists another way, namely, by setting $x = z$ and $j_2 = j_1$. As a result one has a model (*holomorphic and antiholomorphic sectors*) which contains all features of the minimal model coupled to gravity in the Polyakov light-cone gauge. However in contrast to the Polyakov gauge a global structure of 2d world sheet is now well-defined that permits one to compute correlation functions of the physical operators. The latter are given by

$$\mathcal{O}_{n.m} = \int d\mu(x,\bar{x};j_{n.m})\phi_{n.m}(x)\Phi^{j_{n.m}}(\bar{x},x) \ . \tag{9}$$

Here $\mu(x,\bar{x};j_{n.m})$ represents a measure and ϕ, Φ are the primaries of the minimal model and SL(2) degenerate conformal field theory, respectively. As an example, I computed the three point functions of the physical operators $\mathcal{O}_{1.m}$ [11]. The results revealed the same property as was found in the conformal gauge, namely, the OP algebra of the physical operators is not closed anymore [15].

4. Some chiral rings of N=2 discrete superconformal series induced by SL(2) degenerate conformal field theories

In this section I sketch a link between some $N=2$ discrete superconformal series and $SL(2)$ degenerate conformal field theories along the lines of ref.[16].

The starting point is the fermionic construction proposed by Di Vecchia, Petersen, Yu and Zheng to build the unitary representations of the $N=2$ superconformal algebra in terms of free fermions and unitary representations of $\hat{s}l_2$ [17]. In fact one can do better: the only difference between the unitary representations of $\hat{s}l_2$ and degenerate ones is a value of k (see(3)). Therefore one can relate the degenerate representations of $\hat{s}l_2$ to some discrete series of $N=2$. So it allows one to investigate a "minimal" non-unitary sector of the discrete series of $N=2$ (see [16] for more details). As a result, the following relations between conformal dimensions h and U(1) charges q of $N=2$ primaries in the Neveu-Schwarz sector on the one hand and weights j and magnetic quantum numbers μ of $SL(2)$ primaries on the other hand were found

$$h = \frac{j(j+1)}{k+2} - \frac{\mu^2}{k+2} \quad , \quad q = \frac{\mu}{k+2} \quad . \tag{10}$$

In the problem at hand SL(2) primaries are defined as

$$\Phi^j_\mu(z,\bar{z}) = \frac{1}{\mathcal{N}(j,\mu)}\oint_C\oint_{\bar{C}} dxd\bar{x}(x\bar{x})^{\mu-j-1}\Phi^j(x,\bar{x},z,\bar{z}) \ , \tag{11}$$

where C , $\bar{C}$ are closed contours, μ is the magnetic quantum number and $\mathcal{N}(j,\mu)$ - normalization factors [16].

There is also a relation between correlation functions of these theories

$$\langle\prod_{i=1}^N \Phi^{h_i}_{q_i}(z_i,\bar{z}_i)\rangle = \prod_{i<j}^N |z_{ij}|^{\lambda_{ij}} \langle\prod_{i=1}^N \Phi^{j_i}_{\mu_i}(z_i,\bar{z}_i)\rangle \ , \tag{12}$$

with $\lambda_{ij} = -4\mu_i\mu_j/k+2$.

It should be stressed that the primaries fields defined in (11) depend on contours $C(\bar{C})$ in the isotopic spaces. From this point of view one has

the non-local operators. The correct contours $C_i(\bar{C}_i)$, for a particular conformal block, should be chosen by the correct singularities at $z_{ij} \to 0$, which should match to an OP algebra in a consistent way.

Let me restrict to the so-called primary chiral fields[5] [18]. For such fields one has $q = h$. It simplifies integrals over $x(\bar{x})$ and due to this reason one can investigate properties of the OP algebra of the primary chiral fields [16]. It turns out that the fields don't generate the ring. The origin of this disaster is the non-unitarity of the models. In the case at hand the U(1) conservation law doesn't provide a proper selection rule. It forces me to look for more fine structures. In attempting to do this it is advantageous to use operators introduced by Moore and Reshetikhin [19]. The point is that a operator ${}^{\alpha}\Phi^h_q$ is associated with a triple (h, q, α), where h and q are the conformal dimension and U(1) charge. As to α, it means a pair of states in the highest weight representations of the quantum groups $(U_q osp(2/1), U_q sl(2))$. If the states α are the highest weight vectors then the operators ${}^{\alpha}\Phi^h_h$ define the ring [16]. This solution provides a strong evidence that a quantum group underlies the ring. It is disguised in the unitary case in virtue of the U(1) conservation law, but it becomes clear in the non-unitary case.

5. Conclusions and remarks

First, let me say a few words about results.

In the above I have briefly outlined the $SL(2)$ degenerate conformal field theories and their applications to the 2d gravity in the Polyakov light-cone gauge and some $N = 2$ discrete superconformal series. The main moral of the story is the isotopic coordinates $x(\bar{x})$. On the one hand they provide the well-defined OP algebra and enlarge the degree of applications. On the other hand, a natural question arises: is the theory really two-dimensional or it is a restriction of a certain four dimensional one? Unfortunately at this moment I don't know of an exact answer to this magical question.

Let me conclude by mentioning some open problems.

- An important problem which wasn't discussed in [6] is to check that the solutions of the KZ equations also satisfy a system of equations which follows from the singular vectors in the highest weight representations of $\hat{sl}_2$ parametrized by the Kac-Kazhdan set (3).
- The next open problem is to solve non-diagonal $SL(2)$ theories.
- Due to the solution of the $SL(2)$ degenerate conformal field theories, there is a strong indication on a finite number of order parameters in a "parafermionic" theory for a rational k. The problem is to investigate such coset $SL(2)/U(1)$ in more detail. Furthermore there exists another problem, namely, to find models of statistical mechanics which have fixed points described by the coset $SL(2)/U(1)$.
- The main problem in the context of 2d gravity is, of course, to compute four point correlation functions.
- As to the $N = 2$ discrete superconformal theories they are waiting to be solved.

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